JANUARY 2015 ANALYSIS QUALIFYING EXAM

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1. Problem 1

We may cover $\mathbb{C}\backslash\{0\}$ by countably many compact sets K_n. Now, if any K_n had infinitely points of A, we could extract a sequence of distinct elements of A. By compactness, this must have a convergent subsequence; however, by our condition on A, this subsequence must converge to $0 \notin K_n$, which is a contradiction. Then, we deduce that each K_n contains only finitely many points of A, so that A can be written as the countable union of finite sets, and is hence countable, as desired.

2. Problem 2

(a). " $\Longrightarrow$ " Argue by contraposition. If $\#\{n \in \mathbb{N} \mid d(x, x_n) < \epsilon\} < \infty$ for some $\epsilon > 0$, consider

$$\delta := \min_{x_n \in B_\epsilon(x)\backslash\{x\}} \{d(x_n, x)\}$$

This is positive since there are only finitely many positive distances, so that for $\delta/2$, there is no x_n with $d(x_n, x) < \delta/2$, in which case no subsequence can possibly converge.

" $\Longleftarrow$ " For each $k \geqslant 1$, there exists $x_{n_k} \in B_{1/k}(x)$. Choosing x_{n_k} for all k, we see that

$$\lim_{k\to\infty} d(x_{n_k}, x) = 0$$

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so that x_{n_k} is a convergent subsequence.

(b). We use part (a) and argue by contraposition; that is, suppose there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with no accumulation point. We want to show that M is not compact. Then, for each n, there exists ϵ_n such that

$$B_{\epsilon_n}(x_n) \cap M = \{x_n\}$$

Whence $\{x_n\}$ is open for all $n \in \mathbb{N}$. However, every singleton set is closed in a Hausdorff space, in which case we see

$$M = \big(M\backslash(x_n)_{n\in\mathbb{N}}\big) \cup \big(\bigcup_{n=1}^{\infty}\{x_n\}\big)$$

is an open cover that has no finite subcover, so that M is not compact.

3. Problem 3

By Cauchy's integral formula,

$$f'(0) = \frac{1}{2\mathfrak{p}i}\int_{B_r(0)}\frac{f(z)}{z^2}dz$$

Whence

$$2f'(0) = \frac{1}{2\pi i}\int_{B_r(0)}\frac{f(z)-f(-z)}{z^2}dz$$

Taking the modulus of the above,

$$\begin{aligned}2|f'(0)| &\leqslant \frac{1}{2\pi}\int_{B_r(0)}\frac{|f(z)-f(-z)|}{|z|^2}dz\\ &\leqslant \frac{1}{2\pi r^2}\cdot d\cdot 2\pi r = \frac{d}{r}\end{aligned}$$

Letting $r\to 1$, we find

$$2|f'(0)| \leqslant d$$

as contended.

4. Problem 4

We see that if $z^3 + 1 = 0$,

$$z = e^{i\pi/3}, \quad e^{i\pi}, \quad e^{5\pi i/3}$$

By Cauchy's residue formula,

$$\int_\gamma \frac{z(z-2)}{z^3+1} dz = 2\pi i \sum \text{Res}\Big(\frac{z(z-2)}{z^3+1}, z_i\Big)$$

So, computing our residues,

$$\begin{aligned}\lim_{z\to -1} \frac{(z+1)z(z-2)}{z^3+1} &= \lim_{z\to -1} \frac{z(z-2)}{z^2-z+1} \\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{z\to e^{i\pi/3}} \frac{(z+1)z(z-2)}{z^3+1} &= \lim_{z\to e^{i\pi/3}} \frac{z(z-2)}{(z+1)(z-e^{5\pi i/3}} \\ &= \frac{i}{\sqrt{3}} \\ \lim_{z\to e^{5i\pi/3}} \frac{(z+1)z(z-2)}{z^3+1} &= \lim_{z\to e^{5i\pi/3}} \frac{z(z-2)}{(z+1)(z-e^{\pi i/3}} \\ &= \frac{-i}{\sqrt{3}}\end{aligned}$$

In which case

$$\int_\gamma \frac{z(z-2)}{z^3+1} dz = 2\pi i$$

5. Problem 5

If f is integrable, this is trivial by Lebesgue's dominated convergence theorem. Assume then that f is not integrable. We may extract a subsequence f_{n_k} increasing to f, so that by Lebesgue's monotone convergence theorem,

$$\int f_{n_k} d\mu \to \int f = \infty$$

Now, choose an arbitrary subsequence. We may extract a further subsequence that is increasing to f, and by the same logic above, this sub-subsequence must converge to $\int f$. Then, this shows that every subsequence has a further subsequence converging to f, whence $\int f_n \to \int f$.

6. Problem 6

(a). f is absolutely continuous if for every $\epsilon > 0$ there exists δ such that for any set of open intervals $\{(a_k, b_k)\}$ with

$$\sum_{k=1}^{N} b_k - a_k < \delta$$

we have

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

(b). Let $\epsilon > 0$, and suppose A is measurable with $\lambda(A) = 0$. Note first that absolutely continuity ensures the existence of a δ such that any set of open intervals $\{(a_k, b_k)\}$ with $\sum_{k=1}^{N} b_k - a_k < \delta$ implies

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$$

By definition of Lebesgue measure, we may find disjoint open intervals $\{(a_k, b_k)\}_{k\in\mathbb{N}}$ with $A \subset \bigcup_{k=1}^{\infty}(a_k, b_k)$ and

$$\lambda\Big(\bigcup_{k=1}^{\infty}(a_k, b_k)\Big) < \delta$$

Let K be any compact subset of A; we may extract a finite subcover $\{(a_{k_i}, b_{k_i})\}$ of K. Then, we see that

$$\begin{aligned}\lambda(f(K)) &\leqslant \lambda\Big(\bigcup_{i=1}^{N} f(a_{k_i}, b_{k_i})\Big) \\ &\leqslant \sum_{i=1}^{N} |f(b_{k_i}) - f(a_{k_i})| \\ &< \epsilon \quad \text{(Absolute continuity)}\end{aligned}$$

As $\epsilon > 0$ is arbitrary, we deduce that the image of every compact subset of A has measure 0. By continuity and surjectivity of f onto its image, every compact subset of $f(A)$ is the image of some compact subset of A, whence every compact subset of $f(A)$ has zero measure. Inner regularity of Lebesgue measure gives:

$$\begin{aligned}\lambda(f(A)) &= \sup_{K \subset A \text{ cpt}} \{\lambda(K)\} \\ &= 0\end{aligned}$$

So that $\lambda(f(A)) = 0$, as desired.

7. Problem 7

We can show an even stronger result; that is, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.

$$\begin{aligned}\{x \mid \limsup_{n\to\infty} f_n \leqslant c\} &= \bigcap_{k\geqslant 1}\{x \mid \limsup_{n\to\infty} f_n < c + 1/k\} \\ &= \bigcap_{k\geqslant 1}\bigcup_{n\geqslant 1}\bigcap_{m\geqslant n}\{x \mid f_m(x) < c + 1/k\}\end{aligned}$$

Similarly,

$$\begin{aligned}\{x \mid \liminf_{n\to\infty} f_n \leqslant c\} &= \bigcap_{k\geqslant 1}\{x \mid \liminf_{n\to\infty} f_n < c + 1/k\} \\ &= \bigcap_{k\geqslant 1}\bigcap_{n\geqslant 1}\bigcup_{m\geqslant n}\{x \mid f_m(x) < c + 1/k\}\end{aligned}$$

As each f_n is measurable, the above are both clearly measurable sets, so that f is measurable.

8. Problem 8

(a). Recall that $\limsup_{k\to\infty} A_k = \bigcap_{k\geqslant 1}\bigcup_{j\geqslant k} A_j$. In order to interchange the order of the limit and measure, we must verify that at least one term has finite measure. By assumption,

$$\mu(\bigcup_{k\geqslant 1} A_k) \leqslant \sum_{k=1}^{\infty} \mu(A_k) < \infty$$

so that

$$\begin{aligned}\mu(\limsup_{k\to\infty} A_k) &= \lim_{k\to\infty} \mu(\bigcup_{j\geqslant k} A_j) \\ &\leqslant \lim_{k\to\infty} \sum_{j\geqslant k} \mu(A_j) \\ &= 0\end{aligned}$$

so that $\mu(\limsup_{k\to\infty} A_k) = 0$. *For reference, this is commonly referred to as the Borel-Cantelli lemma.*

(b). We define our n_k inductively. Choose n_1 freely. Since f_n is Cauchy in measure, we may find $n_2 > n_1$ such that

$$\mu\big(\{x \mid |f_{n_1}(x) - f_{n_2}(x)| \geqslant 1/2\}\big) < 1/2$$

Now, choose $n_3 > n_2$ such that

$$\mu\big(\{x \mid |f_{n_2}(x) - f_{n_3}(x)| \geqslant 1/4\}\big) < 1/4$$

and, continuing in an inductive fashion, suppose we have chosen n_k satisfying the requirements of the problem, we may choose $n_{k+1} > n_k$ such that

$$\mu\big(\{x \mid |f_{n_k} - f_{n_{k+1}}| \geqslant 1/2^k\}\big) < 1/2^k$$

so that we can construct the subsequence $(n_k)_{k\in\mathbb{N}}$ as desired. Now that $(n_k)_{k\in\mathbb{N}}$ has been chosen, note that

$$\sum_{k=1}^{\infty} \mu(A^{1/2^k}_{n_k,n_{k+1}}) < 1 < \infty$$

So that, applying part (a),

$$\mu(\limsup_{k\to\infty} A^{1/2^k}_{n_k,n_{k+1}}) = 0$$

(c). Suppose $x \in X\backslash A$. Then, by De Morgan's laws,

$$x \in \liminf_{k\to\infty} A^{1/2^k}_{n_k,n_{k+1}}{}^c$$

That is, there exists $K \in \mathbb{N}$ such that for all $k > K$,

$$|f_{n_k} - f_{n_{k+1}}(x)| < \frac{1}{2^k}$$

In particuar, upon fixing x we see that for all $j > k$,

$$\begin{aligned}
|f_{n_k}(x) - f_{n_j}(x)| &\leqslant |f_{n_k}(x) - f_{n_{k+1}}| + |f_{n_{k+1}}(x) - f_{n_{k+2}}| \\
&+ \cdots + |f_{n_{j-1}}(x) - f_{n_j}(x)| \\
&< \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{j-1}} \\
&= \frac{1}{2^k}\Big(\frac{1 - \frac{1}{2^{j-k}}}{1 - \frac{1}{2}}\Big) \\
&< \frac{1}{2^{k-1}}
\end{aligned}$$

Whence we deduce that $(f_{n_k}(x))_{k\in\mathbb{N}}$ is Cauchy. Since this is simply a sequence of real numbers, we use completeness of $\mathbb{R}$ to deduce that $f_{n_k}(x) \to f(x) \in \mathbb{R}$.

(d). By countable subadditivity,

$$\begin{aligned}
\mu(B_m) &\leqslant \mu(A) + \sum_{i\geqslant m} A^{1/2^i}_{n_i,n_{i+1}} \\
&< 0 + \frac{1}{2^{m-1}}
\end{aligned}$$

Letting $m \to \infty$, we find

$$\lim_{m\to\infty} \mu(B_m) = 0$$

as contended.

(e). We simply use the definition of B_m. Note that if $x \notin B_m$, then, $x \notin A$ and $x \in \bigcap_{i\geqslant m} A^{1/2^i}_{n_i,n_{i+1}}$. That is, for all $i \geqslant m$,

$$|f_{n_i}(x) - f_{n_{i+1}}(x)| < \frac{1}{2^i}$$

for all $x \notin B_m$. Then, by an identical computation to part (c), for $i > j \geqslant m$,

$$|f_{n_j}(x) - f_{n_i}(x)|\frac{1}{2^{j-1}}$$

as desired.

(f). Let $\epsilon > 0$. We may find $m \in \mathbb{N}$ such that (by part (e)),

$$\mu\big(\{x \mid |f_{n_m}(x) - f(x)| \geqslant \frac{1}{2^{m-1}}\}\big) < \epsilon$$

To see this more easily, note that since $\mu(B_m) \to 0$. Also, if $x \notin B_m$, we know that for all $k > m$,

$$|f_{n_m}(x) - f_{n_k}(x)| < \frac{1}{2^{m-1}}$$

Letting $k \to \infty$, we find

$$|f_{n_m}(x) - f(x)| < \frac{1}{2^{m-1}}$$

so that $|f_{n_m}(x) - f(x)| \geqslant \frac{1}{2^{m-1}}$ whenever $x \in B_m$, and since $\mu(B_m) \to 0$, there exists $M \in \mathbb{N}$ such that whenever $m > M$, $\mu(B_m) < \epsilon$. Putting this all together, we get the above claim.

(g). Let $\epsilon > 0$. As f_n is Cauchy in measure, there exists $N \in \mathbb{N}$ such that for all n, $m > N$,

$$\mu\big(\{x \mid |f_n(x) - f_m(x)| \geqslant \epsilon/2\}\big) < \frac{\epsilon}{2}$$

And, by part (f), there exists $M \in \mathbb{N}$ such that for all $m > M$,

$$\mu\big(\{x \mid |f_{n_m}(x) - f(x)| \geqslant \epsilon/2\} < \frac{\epsilon}{2}$$

Let $n > N$ and choose $m > M$ such that $n_m > N$. If x is such that

$$|f_n(x) - f(x)| \geqslant \epsilon$$

then,

$$\begin{aligned} 2\epsilon &\leqslant |f_n(x) - f(x)| \\ &\leqslant |f_{n_m}(x) - f(x)| + |f_n(x) - f_{n_m}(x)| \\ \Longrightarrow |f_n(x) - f_{n_m}(x)| &\geqslant \epsilon/2 \quad \text{or} \quad |f_{n_m}(x) - f(x)| \geqslant \epsilon/2 \end{aligned}$$

In which case, by definition,

$$\{x \mid |f_n(x) - f(x)| \geqslant \epsilon\} \subset \{x \mid |f_n(x) - f_{n_m}(x)| \geqslant \epsilon/2\} \cup \{x \mid |f_{n_m}(x) - f(x)| \geqslant \epsilon/2\}$$

Taking measures of the above,

$$\begin{aligned} \mu\big(\{x \mid |f_n(x) - f(x)| \geqslant \epsilon/2\}\big) &\leqslant \mu\big(\{x \mid |f_n(x) - f_{n_m}(x)| \geqslant \epsilon/2\}\big) \\ &\quad + \mu\big(\{x \mid |f_{n_m}(x) - f(x)| \geqslant \epsilon/2\} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

And we deduce that $f_n \to f$ in measure.

9. Problem 9

(a). False. Every sequence is a net, so, consider $\{0,1\}^{\mathbb{N}}$ endowed with the product topology and let $f_n \in \{0,1\}^{2^{\mathbb{N}}}$ be the sequence of functions $f_n : 2^{\mathbb{N}} \to \{0,1\}$ such that

$$f_n(A) = \chi_A(n)$$

If $A \subset \mathbb{N}$ is infinite, choose any other $B \subset A$ such that both B and $A \backslash B$ are infinite. Consider then $\{f_n(B)\}_{n \in A}$. This has no convergent subsequence since by construction, the above is never eventually constant. However, by Tychonoff's theorem, $\{0,1\}^{2^{\mathbb{N}}}$ is compact, so there does exist a convergent subnet of $f_n(B)\}_{n \in A}$; thus, this subnet is clearly not a subsequence since no subsequence can converge.

(b). True. We see

$$\frac{\partial u}{\partial x} = \cos(x)\cosh(y), \quad \frac{\partial v}{\partial y} = \cos(x)\cosh(y)$$

and

$$\frac{\partial u}{\partial y} = \sin(x)\sinh(y), \quad \frac{\partial v}{\partial x} = -\sin(x)\sinh(y)$$

(c). False. Set

$$a_{ij} := \begin{cases} 1, & i = j+1 \\ -1, & i = j-1 \\ 0, & \text{else} \end{cases}$$

Then,

$$\sum_{j=1}^{\infty} a_{ij} = \begin{cases} -1, & i = 1 \\ 0, & \text{else} \end{cases}$$

$$\sum_{i=1}^{\infty} a_{ij} = \begin{cases} 1, & i = 1 \\ 0, & \text{else} \end{cases}$$

So that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = -1 \neq 1 = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

(d). False. The Cantor function is the standard counterexample, as

$$f(1) - f(0) = 1 \neq 0 = \int_0^1 f'(x)dx$$